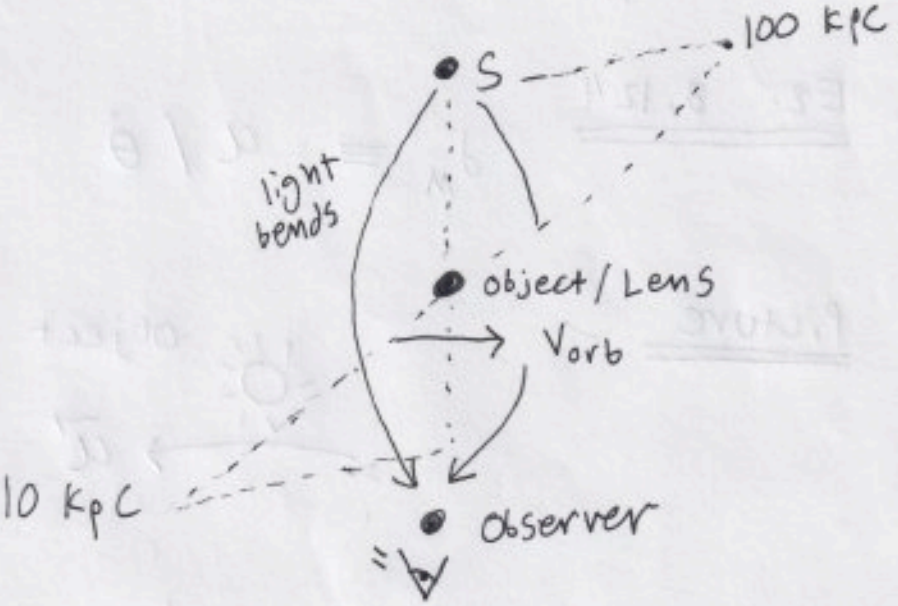
MIT OCW GR PSET 9

1 Timescale for variation in microlensing

- Consider a source that is a star ≈ 100 kPc away and an unseen object of 1 solar mass ≈ 10 kPC away passes in front of it with orbital velocity $v_{orb} \sim 200$ km/sec. What is the timescale over which you expect the light curve to vary?

- This is actually a textbook example that is worked out in full in Hartle's book section 11. We have the formula:

$$t_{var} \approx \frac{\theta_E D_L}{v_{orb}}$$



- for a solar mass 100 kpc away, the einstein angle $\theta_E \approx 10^{-3}\,''$ and $D_L \equiv 10$ kpc

$$\longrightarrow \boxed{t_{var} \sim \frac{(10^{-3\,''})(10\text{ kpc})}{200\text{ km/s}} \approx 0.2\text{ years}}$$

2 Proper Motion Distance

• Using the definition of proper motion distance from Carroll Eq. 8.124, compute $d_M(z)$. Your final result should be similar in form to Eq. 8.123 of Carroll + confirm the rule that:

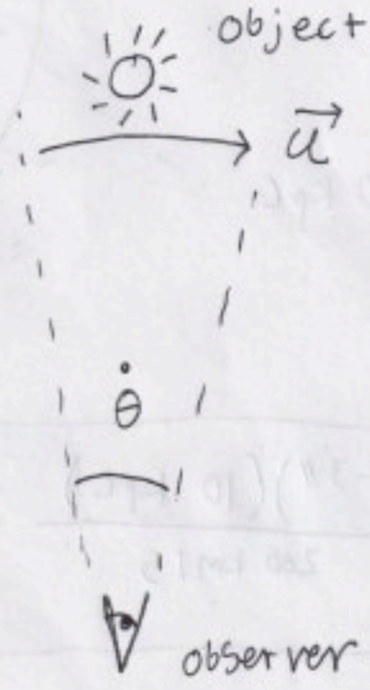
$$d_L(z) = (1+z)\, d_M(z) = (1+z)^2\, d_A(z)$$

where $d_L(z)$ is the "Luminosity Distance" and $d_A(z)$ is "Angular Diameter Distance"

Eq. 8.124

$$d_M \equiv u / \dot{\theta}$$

Picture



• Definition of FRW line element:

$$ds^2 = -dt^2 + a^2(t) R_0^2 \left[d\chi^2 + S_k^2(\chi)\, d\Omega^2 \right]$$

• Consider a null geodesic only moving in the angular direction with $d\chi = 0$:

$$\rightarrow \; 0 = -dt^2 + a^2(t) R_0^2 S_k^2(\chi)\, d\Omega^2$$

• The perpendicular path traveled is given by:

$$d\ell_\perp = a R_0 S_k \, d\Omega$$

$$\rightarrow \; v_\perp = \frac{d\ell_\perp}{dt} = a R_0 S_k \frac{d\Omega}{dt}$$

$$\rightarrow \; d_M \equiv \frac{v_\perp}{\dot{\theta}} = \frac{a R_0 S_k(\chi)\, d\Omega/dt}{d\Omega/dt}$$

$$\rightarrow \; d_M = a(t) R_0 S_k(\chi)$$

• Since only ratios of the scale factor $a(t)$ matter, we are free to set $a(t) = 1$ for "today" (carroll does this in his derivation as well) which implies:

$$d_M = R_0 S_K(\chi)$$

- So all [that's left is to] find $\chi = \chi(z)$ which is the exact same process / result as in Carroll's chapter 8:

$$\rightarrow d_M(z) = \frac{H_0^{-1}}{\sqrt{|\Omega_{c0}|}} S_K\left[\sqrt{|\Omega_{c0}|}\int \frac{dz'}{E(z')}\right]$$

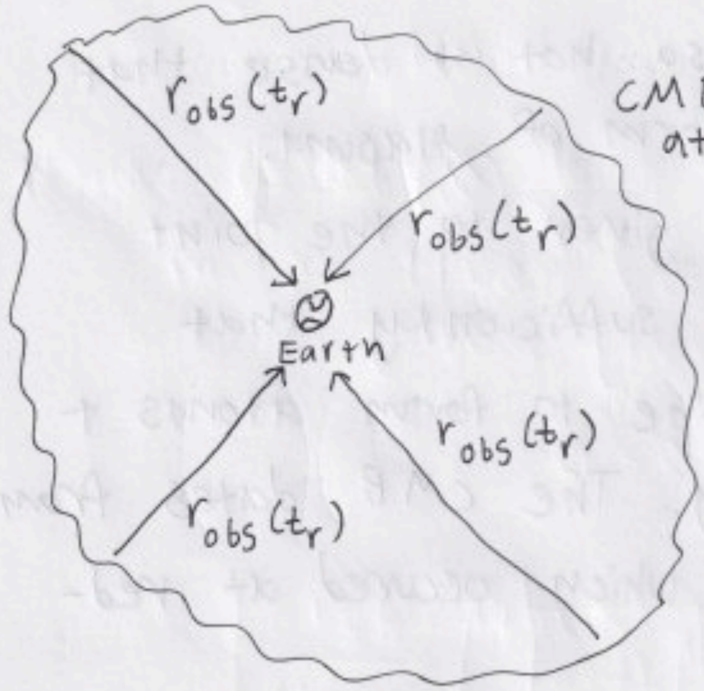
which is exactly 8.123 divided by $(1+z)$ implying $d_M(z) = \dfrac{d_L(z)}{(1+z)}$ as we would expect ✓

$\boxed{3}$ • The early universe was so hot + dense that all matter was in the form of plasma.

"Recombination" is the name given to the point where the universe cooled sufficienty that $\bar{e}$'s and p^+'s could combine to form atoms + photons could stream freely. The CMB dates from the recombination epoch which occured at red-shift $z_r \approx 1200$.

• Consider a $k=0$ FRW cosmology radiation dominated before recombination + matter dominated after. Let $r_H(t_r)$ be the max distance a photon could travel from the Big Bang $(t \equiv 0)$ to recombination $(t \equiv t_r)$. This defines a particle horizon. Only points within the particle horizon can be causally connected since info's max speed is "c".

• Let $r_{obs}(t_r)$ be the max distance a photon could travel from recombination / time of CMB creation to now $t = t_0$ the current epoch. $2 \times r_{obs}(t_r)$ defines the diameter between patches of opposite CMB in our sky centered on Earth.



CMB created at $t = t_r$

$\boxed{a}$ · compute $r_H(t_r)$ & $r_{obs}(t_r)$

· Begin with $r_H(t_r)$... from $t = 0$ to $t = t_r$ the universe was radiation dominated.

· This means $a \propto t^{1/2}$ where "a" is the $a(t)$ from the FRW metric with $k=0$:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and the Hubble parameter:

$$H(t) = \dot{a}(t)/a(t)$$

· Qualitatively, $a(t)$ is related to the rate of stretching / inflation of the universe:

- $r_H(t_r)$ represents the characteristic path length light could have traveled from $t=0$ to $t=t_r$...
- since dimensions of $[a(t)] = \frac{\text{time}}{\text{length}}$ then

$dt/a(t)$ represents an infinitessimal length step dr_H and the total characteristic path length is:

$$r_H(t_r) = \int_0^{t_r} \frac{dt}{a(t)} \quad ; \quad a_{rad}(t) = c_{rad}\, t^{1/2}$$

$\hookleftarrow$ rad. dominated

$$\rightarrow r_H(t_r) = 2c_{rad}^{-1}\, t_r^{1/2}$$

- and define $H_r \equiv H(t_r) = \dfrac{\frac{1}{2} c_{rad}\, t_r^{-1/2}}{c_{rad}\, t_r^{1/2}} = \dfrac{1}{2t_r}$ ☆

as well as $a_r \equiv a(t_r) = c_{rad}\, t_r^{1/2}$

$$\rightarrow c_{rad} = a_r / t_r^{1/2}$$

$$\rightarrow \boxed{r_H(t_r) = 2t_r / a_r}$$

- During the time t_r to t_0 (now) the universe is matter dominated meaning:

$$a(t) = c_m t^{2/3} \propto t^{2/3}$$

· Performing the same calculation we get:

$$r_{obs}(t_r) = \int_{t_r}^{t_0} \frac{dt}{a_m(t)} = \int_{t_r}^{t_0} \frac{dt}{c_m t^{2/3}}$$

$$= \left(\frac{3}{c_m}\right)\left(t_0^{1/3} - t_r^{1/3}\right)$$

· But we can again use our definition:

$$a(t_r) = a_r = c_m t_r^{2/3}$$

$$\rightarrow \boxed{r_{obs}(t_r) = \left(\frac{3t_r^{2/3}}{a_r}\right)\left(t_0^{1/3} - t_r^{1/3}\right)}$$

· The CMB is extremely isotropic...

$\boxed{b}$ · Given the isotropy of the CMB why is the fact that $r_H / r_{obs} \ll 1$ puzzling?

· First let's confirm that $r_H / r_{obs} \ll 1$:

$$r_H / r_{obs} = \left(\frac{\cancel{a_r}}{3t_r^{2/3}}\right)\left(\frac{2t_r}{\cancel{a_r}}\right)\left(t_0^{1/3} - t_r^{1/3}\right)^{-1}$$

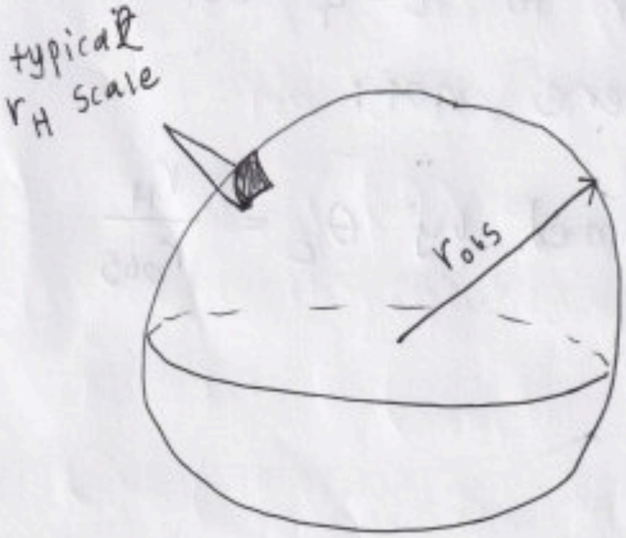
$$= \frac{2t_r^{1/3}}{3(t_0^{1/3} - t_r^{1/3})}$$ and $t_r \ll t_0$ since

· The CMB happened a very long time ago:

$$\rightarrow r_H / r_{obs} \approx \left(\frac{2}{3}\right)\left(\frac{t_r}{t_0}\right)^{1/3} \approx \frac{2}{3}\alpha^{1/3} \quad \text{where } \alpha \ll 1$$

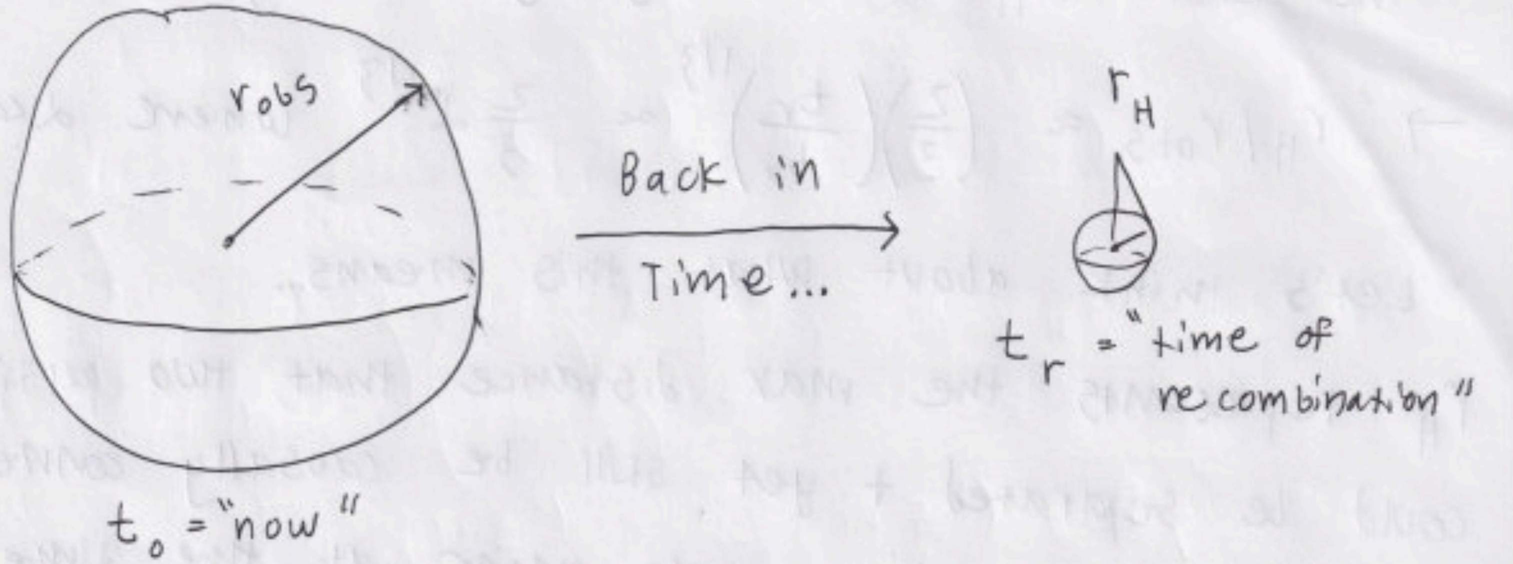
· Let's think about what this means...

r_H represents the max distance that two particles could be separated + yet still be causally connected / in equilibrium with each other at the time of the CMB creation / recombination. However this distance is much smaller than the current scale of the CMB:



· So we would expect that only tiny splotches the size of r_H in the CMB could be homogeneous but we see the entire thing is extremely homogeneous

· In order to avoid Faster Than Light Travel, the solution then is that the CMB / size of the universe must have originally been much smaller + order $\sim r_H$:



- so this implies the universe is expanding $\ddot{O}$!!

$\boxed{c}$ • compute the angular scale θ_c such that sources separated by $\theta < \theta_c$ as seen today were in causal contact prior to $t = t_r$ but sources separated by $\theta > \theta_c$ were not:

- This is just the angle defined by $\theta_c = \frac{r_H}{r_{obs}}$

$$= \frac{2 t_r^{1/3}}{3\left(t_0^{1/3} - t_r^{1/3}\right)}$$

- We can rewrite this in terms of the Hubble parameter with the following formulas from wikipedia:

$$\frac{H(z)}{H_0} = \sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$$

is the Hubble parameter in terms of redshift "z" where the Ω's are dimensionless weight parameters determining the makeup of matter/spacetime in our universe with:

$$\Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda = 1$$

radiation param; matter weight; curvature weight; weight associated w/ cosmological constant

- for a matter dominated universe, $\Omega_m \to 1$

and $H(z) \approx H_0 (1+z)^{3/2}$

$\rightarrow H_r = H(z_r \approx 1200) \approx H_0 (1200)^{3/2} \approx \frac{1}{2t_r}$

from ☆

$\rightarrow t_r \approx \frac{1}{2} H_0^{-1} (1200)^{-3/2}$

Also $H_0 = H(t = t_0) \rightarrow \frac{2}{3t_0}$ analogous to ☆ with matter domination $a \propto t^{2/3}$

$$\rightarrow \theta_c = \left(\frac{2}{3}\right) \frac{\left(0.5^{1/3} H_0^{-1/3} (1200)^{-1/2}\right)}{\left(0.67^{1/3} H_0^{-1/3} - 0.5^{1/3} H_0^{-1/3} (1200)^{-1/2}\right)}$$

$\rightarrow$ $\theta_c \approx 0.017$ radians or convert to degrees by $180°/\pi$ radians

$\rightarrow$ $\boxed{\theta_c \approx 1°}$ • So we would only expect patches of our sky within $1°$ of each other to be within equilibrium in the CMB, but the entire CMB in our sky is homogeneous again leading to the inflation theory ✓

$\boxed{4}$ • Suppose that at some very early epoch, $t_1 \leq t \leq t_2$ where $t_1 \ll t_2 \ll t_r$ the universe resides in a "false vacuum" state: A scalar field ϕ fills spacetime + provides an effective vacuum energy $\rho_{vac} \approx V(\phi)$. The vacuum energy acts like a cosmological constant $\Lambda = 8\pi \rho_{vac}$. You may assume ρ_{vac} is like a gas of photons (even though it isn't) s.t.

$\rho_{vac} \approx \frac{4\sigma}{c} T_{vac}^4$ where $k_B T_{vac} \approx 10^{15}$ GeV.

• As the universe expands, the potential $V(\phi)$ slowly evolves. Around t_2, the scalar field decays into standard model particles, + the stress-energy tensor is no longer dominated by $V(\phi)$. These particles provide the matter + radiation content for our universe; it is then radiation dominated until recombination, and matter dominated after...

$\boxed{a}$ Compute the value of the effective cosmological constant Λ in sec^{-2} w/ the fact that 1 MeV $= 1.6 \times 10^{-12}$ gm $\cdot$ cm^2/sec^2 :

$$\Lambda = 8\pi \rho_{vac} = \frac{32\pi\sigma}{c} T_{vac}^4$$

$$= \frac{32\pi\sigma\,(k_B T_{vac})^4}{c k_B^4} \quad ; \; k_B \approx 8.6 \times 10^{-11} \text{ MeV/K}$$

$$k_B T_{vac} \approx 10^{15} \text{ GeV}$$

$$\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3}$$

$$\rightarrow \Lambda \approx \frac{32\pi^3}{60 c^3 \hbar^3} 10^{60} (\text{GeV})^4$$

$$\hbar \approx 6.6 \times 10^{-22} \text{ MeV} \cdot \text{sec}$$

$$\rightarrow \Lambda \approx \left(\frac{8\pi^3}{15 c^3}\right) \frac{10^{60} \cdot 1000^4 \, (\text{MeV})^4}{(6.6)^3 \times 10^{-66} \, (\text{MeV})^3 (\text{sec})^3}$$

$$\rightarrow \Lambda \approx \left(\frac{8\pi^3}{15c^3}\right)\left(\frac{10^{138}}{6.6^3}\right)\frac{\text{MeV}}{\text{sec}^3}$$

and $1 \text{ MeV} \approx 1.6 \times 10^{-12} \frac{\text{gm}\cdot\text{cm}^2}{\text{sec}^2}$

$$\rightarrow \Lambda \approx \left(\frac{8\pi^3}{15c^3}\right)\left(\frac{1.6}{6.6^3}\right)\left(10^{126}\right)\frac{\text{gm}\cdot\text{cm}^2}{\text{sec}^5}$$

$$c = 3\times10^8 \text{ m/s} = 3\times10^{10} \text{ cm/sec}$$

$$\rightarrow c^3 = 27\times10^{30} \text{ cm}^3/\text{sec}^3$$

$$\rightarrow \Lambda \approx \left(\frac{8\pi^3}{15}\right)\left(\frac{1.6}{(27\times6.6^3)}\right)\left(10^{96}\right)\frac{\text{gm}}{\text{cm}\cdot\text{sec}^2}$$

$$G = 6.67\times10^{-8} \text{ cm}^3\cdot\text{gm}^{-1}\cdot\text{sec}^{-2}$$

• to get Λ in units of sec^{-2} must multiply by Gc^{-2} :

$$\Lambda = \left(\frac{8\pi^3}{15}\right)\left(\frac{1.6\times10^{96}}{27\times6.6^3}\right)\times\left(\frac{6.67\times10^{-8}}{9\times10^{20}}\right)\text{sec}^{-2}$$

$$\boxed{\Lambda \approx 2.53\times10^{65} \text{ sec}^{-2}}$$

[b] repeat the calculation from 3.a. where you found $r_H(t_r)$ and $r_{obs}(t_r)$. Find $N_e \equiv \Delta t \sqrt{\Lambda/3}$ where $\Delta t = t_2 - t_1$ which forces $r_H / r_{obs} = r_{obs}/r_H = 1$. N_e is called the number of e-foldings. Estimate t_2:

- Begin with the 1st Fridmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \quad \text{but } k=0 \text{ for}$$

this problem / assumption and we are given that $\Lambda = 8\pi\rho$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}(G+1) \approx \frac{\Lambda}{3}$$

$$\rightarrow \dot{a} = \sqrt{\Lambda/3}\, a \rightarrow a(t) = a_{initial} \exp\Big\{\overbrace{\sqrt{\Lambda/3}\, t}^{\overset{H_{initial}\, t}{\equiv}}\Big\}$$

- So let's calculate the particle horizon / mean free path during this inflationary period:

$$r_{inflation} = \int_0^{\Delta t} \frac{dt}{a(t)} = \left(\frac{1}{a_{init}}\right)\left(\frac{-1}{H_{init}}\right) e^{-H_{init} t}\Big|_0^{\Delta t}$$

$$\rightarrow r_{inflation} = \left(\frac{-1}{a_{init} H_{init}}\right)\left(\underbrace{e^{-H_{init} \Delta t}}_{\approx 0} - 1\right)$$

$$\rightarrow r_{inflation} \approx 1/a_{init} H_{init}$$

. Now rather than assuming an intermediary radiation dominated period followed by a matter dominated period; just assume from Δt to t_0 (our current time now) that the universe has been radiation dominated:

$$\rightarrow a(t) \propto t^{1/2}$$

$$r_{obs} = \int_{\Delta t}^{t_0} \frac{dt}{c_r t^{1/2}} = \frac{2}{c_r} t^{1/2} \Big|_{\Delta t}^{t_0}$$

$$a_0 = a(t = t_0) = c_r t_0^{1/2}$$

$$\dot{a}/a = H(t) = \tfrac{1}{2} c_r t^{-1/2} / c_r t^{1/2} = \frac{1}{2t}$$

$$\rightarrow t = \frac{1}{2H} \quad \text{after inflation} \quad ★$$

$$\rightarrow r_{obs} = \left(\frac{2 t_0^{1/2}}{a_0}\right)\left(t_0^{1/2} - \underbrace{(\Delta t)^{1/2}}_{\approx 0}\right) \approx \frac{2 t_0}{a_0}$$

$\rightarrow r_{obs} \approx 1/H_0 a_0$

$\rightarrow r_{obs}/r_{inflation} = r_{inflation}/r_{obs} = 1$ by construction

$$\rightarrow \frac{a_{init} H_{init}}{H_0 a_0} = 1$$

- assume H is constant from the beginning to end of the inflationary period s.t. H_{init} = constant
- However, we know $a(t)$ is exponential in time through the inflationary period s.t. $a_{end} = e^N a_{init}$ where a_{end} is the $a(t)$ at the end of inflaxion

$$\rightarrow \frac{e^{-N} a_{end} H_{init}}{a_0 \ H_0} = 1$$

[redacted] · since the universe is radiation dominated from a_{end} to a_0 (by assumption) $\rightarrow a \propto t^{1/2} \propto H^{-1/2}$

[redacted] via ☆

$$\rightarrow e^{-N} \left(\frac{H_{init}^{-1/2} \ H_{init}}{H_0^{-1/2} \ H_0} \right) = 1$$

$$\rightarrow e^N = \left(\frac{H_{init}}{H_0}\right)^{1/2}$$

$$\rightarrow \ln(e^N) = \ln\left((H_{init}/H_0)^{1/2}\right)$$

$$\rightarrow N_e = \frac{1}{2}\ln(H_{init}/H_0)$$

H_{init} + H_0 are both things we can look up / approximate. H_0 can be looked up + is of the order 10^{-42} GeV. For H_{init}, we look for the energy scale during the inflationary period. The problem states this is 10^{15} GeV

$$\rightarrow \boxed{N_e = \frac{1}{2}\ln(10^{57}) \approx 64}$$

- Throughout this problem we could have made other approximations, etc. but they would all lead to the number of "e-foldings" being about ~60 ...

- Assuming $t_1 \approx 0 \rightarrow \Delta t \approx t_2$

$$\rightarrow 64 \approx N_e \approx t_2(\Lambda/3)^{1/2} \approx H_{init}\, t_2$$

$t_2 \approx 64 \times 10^{-15}\ \mathrm{GeV}^{-1}$

can convert to seconds if you like

$\boxed{c}$ · What is the spatial expansion factor $a(t_2)/a(t_1)$ during this inflationary epoch?

$$\frac{a(t_2)}{a(t_1)} = \frac{a_{init} \exp\{H_{init} t_2\}}{a_{init} \exp\{\approx 0\}} \approx e^{H_{init} t_2}$$

$$\rightarrow \boxed{\frac{a(t_2)}{a(t_1)} \approx e^{N_e} = e^{64}}$$

$\boxed{d}$ · Recalculate the angular scale θ_c in this inflationary universe:

· We can infer this based on our physical intuition... since $r_H = r_{obs}$ now, the points on opposite sides of the CMB should be in causal contact, so

$$\boxed{\theta_c = 180°}$$

$\boxed{5}$ The flatness problem

$\boxed{a}$ · Use the Friedmann equations to derive a general expression relating values of Ω at different times - i.e. Ω_1 at t_1 (corresponding to a scale factor (a_1) and Ω_2 at t_2 (corresponding to a_2). Do this calculation both for a matter and radiation dominated universes. Express your answer in terms of $\Omega - 1$ and the scale factor. Do not take into account the inflationary physics of problem 4:

$$(F1) \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$(F2) \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

• We define $\rho_{crit} \equiv 3H^2/8\pi G$

and $\Omega = \rho/\rho_{crit}$ and $\dot{a}/a = H$

$$\rightarrow \rho_{crit} = \rho - \frac{3k}{8\pi G a^2} + \frac{\Lambda}{8\pi G}$$

$$\rightarrow \Omega = \frac{\rho}{\rho - \frac{3k}{8\pi G a^2} + \frac{\Lambda}{8\pi G}}$$

$$\rightarrow \Omega - 1 = \frac{\rho - \rho + \frac{3k}{8\pi G a^2} - \frac{\Lambda}{8\pi G}}{\rho - \frac{3k}{8\pi G a^2} + \frac{\Lambda}{8\pi G}}$$

$$\rightarrow \Omega - 1 = \frac{3k - \Lambda a^2}{8\pi G \rho a^2 - 3k + \Lambda a^2}$$

· Again using F1; $8\pi G \rho a^2 = 3a^2\left(H^2 + \frac{k}{a^2} - \frac{\Lambda}{3}\right)$

$$= 3a^2H^2 + 3k - \Lambda a^2$$

$$\rightarrow \Omega - 1 = \frac{3k - \Lambda a^2}{3a^2H^2 + \cancel{3k} - \cancel{\Lambda a^2} - \cancel{3k} + \cancel{\Lambda a^2}}$$

$$\Omega - 1 = \frac{3k - \Lambda a^2}{3a^2H^2}$$

· Now here is where we start approximating; the energy density ρ_R for a radiation dominated universe is $\rho_R \propto a^{-4}$; ρ_m for a matter dominated universe is $\rho_m \propto a^{-3}$; & for the cosmological constant ρ_Λ = constant. So ρ_R dominates early on in the universe (small $a(t)$) then ρ_m dominates, & then ρ_Λ late in our universe's life-time. Assuming we are still in the matter / radiation dominated stages this implies that...

$$\boxed{\Omega - 1 \approx k^{*}/a^2H^2}$$ (☆)

- as the Prof. had pointed out in lecture... ✓

[b] • Observations today tell us that the universe is flat with about 1% accuracy s.t. $\Omega = 1 \pm 0.01$

• Continuing to neglect inflation, estimate the value of $\Omega - 1$ at the epoch of nucleosynthesis ($T_N \sim 1$ MeV) and at the presumed epoch of grand unification ($T_{GUT} \sim 10^{15}$ GeV). Given that the universe is flat with at least 1% accuracy today, how close to flat must it have been in these earlier epochs?

• Before beginning this part (b); let's go further with (☆). For a radiation dominated universe;

$a \propto t^{1/2} \rightarrow H \propto t^{-1} \rightarrow H \propto a^{-2}$

• For a matter dominated universe;

$a \propto t^{2/3} \rightarrow H \propto t^{-1} \rightarrow H \propto a^{-3/2}$

• Now define Ω_2 at a_2 and Ω_1 at a_1:

· for radiation domination:

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = \frac{a_1^2 H_1^2}{a_2^2 H_2^2} = \frac{a_1^2 a_1^{-4}}{a_2^2 a_2^{-4}} = \frac{a_2^2}{a_1^2}$$

$$\rightarrow \boxed{\frac{\Omega_2 - 1}{\Omega_1 - 1} = \left(\frac{a_2}{a_1}\right)^2} \quad (i)$$

· for matter domination:

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = \frac{a_1^2 H_1^2}{a_2^2 H_2^2} = \frac{a_1^2 a_1^{-3}}{a_2^2 a_2^{-3}} = \frac{a_2}{a_1}$$

$$\rightarrow \boxed{\frac{\Omega_2 - 1}{\Omega_1 - 1} = \left(\frac{a_2}{a_1}\right)} \quad (ii)$$

· Now these equations will be more useful to us in part $\boxed{b}$ since they define the time-evolution/ dynamics of $\Omega - 1$ during our current epoch back to nucleosynthesis (the matter dominated dynamic eq. (ii)) and from nucleosynthesis back to grand unification ... since the problem gives us values for temperature during these epochs rather than values for $a(t)$; we will need to convert using the footnote that:

$$T_{past} = \frac{T_{now} \cdot a_{past}}{a_{now}}$$

- First use (ii) to work backwards from current day to nucleosynthesis:

$$\frac{\Omega_N - 1}{\Omega_0 - 1} = \frac{a_N}{a_0} = \frac{T_N}{T_0}$$

- The PSET states $T_0 \approx 2.73\ K \rightarrow 2.35 \times 10^{-10}$ MeV and $T_N \approx 1$ MeV
- <u>Also:</u> $\Omega_0 - 1 \approx 0.01 = 10^{-2}$

$$\rightarrow (\Omega_N - 1) = \left(\frac{1\ \text{MeV}}{10^{-10}\,\text{MeV}}\right) 10^{-2}$$

$$\rightarrow \boxed{(\Omega_N - 1) \approx 10^8}$$

- Now work backwards from nucleosynthesis to grand unification using (i):

$$\frac{\Omega_{GUT} - 1}{\Omega_N - 1} = \left(\frac{a_{GUT}}{a_N}\right)^2 = \left(\frac{T_{GUT}}{T_N}\right)^2$$

• And we are given that $T_{GUT} \sim 10^{15}$ GeV

$\rightarrow T_{GUT} \sim 10^{18}$ MeV

$$\rightarrow (\Omega_{GUT} - 1) \approx \left(\frac{10^{18} \text{ MeV}}{1 \text{ MeV}}\right)^2 10^8$$

$$\rightarrow \boxed{(\Omega_{GUT} - 1) \approx 10^{44}}$$

• Therefore; if $\Omega_0 - 1 \approx 10^{-2}$ implies a very flat universe today; these calculations show that the universe was incredibly <u>NOT</u> flat early on with increasing curvature as you go back in time...

[c] • Repeat the derivation of these omega time-evolution equations using the inflationary universe of problem 4:

• We begin with our result that:

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = \frac{a_1^2 H_1^2}{a_2^2 H_2^2}$$ which made no assumptions on the type of growth of $a(t)$:

• We know during inflation $a(t) \propto e^{Ht}$ where we assume H is constant during all of inflation. We know N e-foldings happen from the beginning to end of inflation so we have:

$$a_{end} = a_{start}\, e^{N}$$

• Since $H = \text{const}$ during inflation we can cancel it out of our Ω expression to get:

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = \frac{a_1^2}{a_2^2} \quad \text{now let} \left\{ \begin{array}{l} 2 \to \text{"end"} \\ 1 \to \text{"start"} \end{array} \right\}$$

$$\rightarrow \boxed{\frac{\Omega_{end} - 1}{\Omega_{start} - 1} = \frac{a_{start}^2}{e^{2N} a_{start}^2} = e^{-2N} \approx e^{-128} \quad \text{since } N \approx 64}$$

D Assuming $\Omega_{start} - 1 \approx 10^{44}$ as we found in the last part we get that:

$$\boxed{(\Omega_{end} - 1) \approx (e^{-128})(10^{44}) \ll 0.01}$$

• So inflation solves the flatness problem!!